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13. ABSTRACT *(Maximum 200 words)*

Asymptotic problems for classical stochastic processes and PDE's, leading to processes and boundary problems on graphs and on similar multi-dimensional spaces, were considered. Perturbations of Hamiltonian systems is an example of such problems. Wave fronts in reaction-diffusion systems are studied using the large deviations theory. Homogenization for RDE's was studied. Asymptotic image reconstruction problems were considered.

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"PDE's, Random Processes and Fields: Asymptotic Problems"

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Several classes of asymptotic problems for stochastic processes and partial differential equations were studied under this Grant:

1. Asymptotic problems for classical processes and PDE's leading to processes on graphs: random perturbations of dynamical systems, processes and PDE's in narrow tubes;

2. Wave fronts and other patterns in reaction diffusion equations;

3. Perturbations of PDE's, homogenization;

4. Asymptotic statistical problems in image reconstruction.

1 Asymptotic problems leading to processes on graphs

The white noise perturbations of Hamiltonian systems on the plane were studied in [2]:

$$\dot{X}_t^\epsilon = \overline{\nabla} H(X_t^\epsilon) + \sqrt{\epsilon}\dot{W}_t, \quad X_0^\epsilon = x \in R^2. \tag{1}$$

Here W_t is the Brownian motion in $R^2, \overline{\nabla} H(x) = (\frac{\partial H}{\partial X^2}(x), -\frac{\partial H}{\partial x^1})$, $0 < \epsilon << 1$. Since $H(x)$ is a first integral for the non-perturbed system (1), $H(X_t^\epsilon)$ is changing slowly as $\epsilon << 1$. In the simplest case, when $H(x)$ has just one minimum and $H(x) \to \infty$ as $|x| \to \infty$, the slow component in a proper time scale converges weakly to an one-dimensional Markov process: $X_{t/\epsilon}^\epsilon \to Y_t$. The Markov character of the limiting process allows to calculate the asymptotics of many interesting characteristics of the process X_t^ϵ. In the general case, when $H(x)$ has many critical points, $H(X_{t/\epsilon}^\epsilon)$ does not converge to a Markov process. A breakthrough in this problem was made in [2]. A graph Γ in the general case can be connected with the Hamiltonian $H(x)$. This graph "counts" the connected components of the level sets of $H(x)$. Let $Y : R^2 \to \Gamma$ be the mapping such that $Y(x), x \in R^2$, is the

point of Γ corresponding to the connected component of the level set of $H(x)L$ containing x. It is proved in [2] that the process $Y(X^{\epsilon}_{t/\epsilon})$ on the graph Γ converge weakly in the space of continuous functions on $[0,T]$ with values in Γ to a diffusion process Y_t on Γ. Inside the edges of Γ the process Y_t is defined by an averaging principle. The behavior at the vertices is described by special gluing correlations,which are expressed through the Hamiltonian.

A similar result for fast oscillating random perturbations of dynamical systems is obtained in [9]. Systems of the form

$$\dot{X}^{\epsilon}_t = b(X^{\epsilon}_t, \xi_{t/\epsilon}), \quad X^{\epsilon}_0 = x \in R^r,$$

where ξ_t is a stationary process with good enough mixing properties, $0 < \epsilon << 1$, were considered in [9]. If the averaged system

$$\dot{\bar{X}}_t = \bar{b}(\bar{X}_t), \quad \bar{b}(x) = Eb(x_1\xi_t), \bar{X}_t = x, \tag{2}$$

has a first integral $\mathcal{A}(x)$, then, under certain assumptions, the evolution of the slow component should be considered on a graph connected with $\mathcal{A}(x)$. If the system (2) has $\ell > 1$ first integrals, the graph should be replaced by a phase space consisting of glued ℓ-dimensional pieces.

A number of generalizations of these problems considered in [4]. In particular, perturbations of the area-preserving systems on two-dimensional torus is considered there. In the last case the limiting process on the graph will have a delay at one of the vertices. Perturbations of the Markov processes with the conservation laws were also considered in [4].

A number of interesting results concerning the boundary problems for second order linear elliptic differential equations follow from [1], [2], [9]. These results are related to behavior of the solutions of the Navier-Stocks equations for the large Reynolds number.

An optimal stabilization problem for a dynamical system with a conservation law, perturbed by a small white noise, is studied in [11]. The approach developed in [2] allows to give an explicit description of a control which is not worse than any other control if the noise is small enough.

It turns our that the processes on graphs and on multidimensional generalizations of graphs arise in many asymptotic problems. Some other examples of "classical" problems, leading to processes on graphs, considered in [1], [5]. In particular, the boundary problems and diffusion processes on narrow tubes were studied there (these results and their development are mentioned in the next section in connections with RDE's). The initial-boundary problems for a class of PDE systems with a small parameter lead to corresponding problems on graphs.

There are many interesting and important for applications problems in this area, especially problems concerning systems with several smooth conservation laws.

2 Asymptotic problems for RDE's

Let Γ be a graph imbedded in R^r and $G^\epsilon be$ the domain consisting of the union of certain neighborhoods of the edges and the vertices included in Γ. Assume that G^ϵ converges Γ as $\epsilon \downarrow 0$. Consider a system of RDE's in $G^\epsilon \times [0, \infty)$:

$$(3) \qquad \begin{aligned} &\frac{\partial u_k^\epsilon(t,x)}{\partial t} = \frac{D}{2} k \Delta u_k^\epsilon + f_k(x; u_1^\epsilon, \dots, u_n^\epsilon), \quad t > 0, x \in G^\epsilon, \\ &\frac{\partial u_k^\epsilon(t,k)}{\partial n}\Big|_{x \in \partial G^\epsilon} = 0, u_k^\epsilon(0,x) = g_k(x), \ k = 1, \cdots, n. \end{aligned}$$

Such problems arise in the combustion theory, in nerve impulse propagation models and other applications. One can expect that the solution of (3) converges to the solution of a proper problem on Γ as $\epsilon \downarrow 0$. Under certain conditions, one can write down the equation for the limiting function of Γ inside each edge of the graph. The initial functions for the limiting problem coincide with the traces of the functions $g_k(x)$ on Γ. But such an initial problem on Γ has no uniqueness : Proper gluing conditions in the vertices should be added. This problem was considered in [1], [5]. The equations inside the edges and the gluing conditions were calculated there. The gluing conditions at a vertex 0 essentially depend on the relation of the order of the size of the neighborhood of 0 in G^ϵ as $\epsilon \downarrow 0$ and the width of the "tubes" connected with 0. The limiting process on Γ can have delay at 0. The vertex 0 under certain conditions can serve as a trap for the diffusing particle: the particle stays forever at 0 after hitting this vertex. All those types of behavior of the limiting process lead to various gluing conditions and effect the behavior of the solution of problem (3) as $\epsilon << 1$. These results also allow to study the wave propagation problem in domains consisting of narrow tubes.

The Dirichlet problem for the linear reaction-diffusion system with a small parameter is studied in [7]:

$$(4) \qquad \begin{aligned} &\frac{\epsilon}{2} \sum_{i,j=1}^{r} a_k^{ij}(x) \frac{\partial^2 u_k^\epsilon}{\partial x^i \partial x^j} + \sum_{i=1}^{r} b_k^i(x) \frac{\partial u_k^\epsilon}{\partial x^i} + \frac{1}{\epsilon} \sum_{j=1}^{n} c_{kj}(x)(u_j(x) - u_k(x)) = 0, \\ &x \in G \subset R^r; \ k = 1, 2, \dots, n; \ u_k(x)\Big|_{\partial G} = \psi_k(x). \end{aligned}$$

Here $c_{kj}(x) \geq 0$, $0 < \epsilon << 1$. Let $(q_1(x), \dots, q_n(x))$ be the stationary distribution for the Markov process in the finite phase space $\{1, \dots, n\}$ with the transition intensities $c_{ij}(x); \bar{b}(x) = \sum_{k=1}^n c_k(x) b_k(x)$. The trajectories $\bar{X}_t$ of the vector field $\bar{b}(x)$ play an important part in the limit behavior of $(u_1^\epsilon(x), \dots, u_n^\epsilon(x))$ as $\epsilon \downarrow 0$. If $\bar{X}_t$ leave G in a finite time starting from any $x \in G$, we have a counterpart of so called Levinson's case for one equation. If $\bar{X}_t$ has an attractor inside G, we have a counterpart of the large deviation case. These cases and some other considered in [7]. A number of new, in comparison with the single equation case, effects appear for the systems. For example, values of the boundary functions at the non-regular for the field $\bar{b}(x)$ part of ∂G can influence the limit of

$u_k^\epsilon(x)$ even in the Levinson case. The large deviation principle for the stochastic processes $(X_t^\epsilon, \nu_t^\epsilon)$ in $R^r \times \{1, \dots, n\}$ corresponding to the system (4) plays the main part in [7].

The same large deviation principle allows to study the wave front propagation for a general RDE-system of Kolmogorov-Petrovskii-Piskunov type:

$$\frac{\partial_k^\epsilon(t,k)}{\partial t} = \frac{\epsilon}{2}\sum_{i,j=1}^{r} a_k^{ij}(x)\frac{\partial^2 u^\epsilon}{\partial x^i \partial x^j} + \frac{1}{\epsilon}F_k(x_j u_1^\epsilon, \dots, u_n^\epsilon)$$

$$u_k^\epsilon(0,x) = g_k(x) \geq 0,\ x \in R^r,\ t > 0,\ k \in \{1, \dots, n\}, \tag{5}$$

$$0 < \epsilon << 1.$$

We assume that a cube $[0, B]^n \subset R^n$ is an invariant domain for the flow corresponding to the vector field $F(x,u) = (F_1(x,u), \dots, F_n(x,u))$ for any fixed $x \in R^r$, that the origin is an unstable point of this flow, and that a counterpart of KPP condition for a single equation holds. Under these conditions, we introduce a non-positive function $V(t,x)$ which is defined by the coefficients $a_k^{ij}(x)$, the nonlinear terms $F_k(x,u)$ and the support of $\sum_{k=1}^n g_k(x)$ such that

$$\lim_{\epsilon \downarrow 0} u_k^\epsilon(t,x) = 0, \quad \text{if} \quad V(t,x) < 0,$$

$$\lim_{\epsilon \downarrow 0} u_k^\epsilon(t,x) > 0, \quad \text{if} \quad (t,x) \in (\{V(t,x) = 0\}),$$

where $(\{V(t,x) = 0\})$ means the interior of the set $\{(t,x)\} : V(t,x) = 0\}$. Function $V(t,x)$ is expressed through the action functional for the large deviations connected with the process $(X_t^\epsilon, \nu_t^\epsilon)$ and through the functions $C_{kj}(x) = \left.\frac{\partial F_k(x,u)}{\partial u_j}\right|_{u=0}$. These results and examples of their applications are considered in [8]. Another approach to the patterns formation problems got RDE's, so called large scale approximation for RDE's, suggested in [5]. This approach is also based on the limit theorems for large deviations.

3 Perturbations of PDE's. Homogenization.

Consider a second order elliptic differential operator L^ϵ depending on a parameter $\epsilon > 0$. Let X_t^ϵ be the diffusion process in R^r corresponding to L^ϵ. There are many results in the theory of differential equations providing sufficient conditions for convergence of solutions of initial boundary problems connected with L^ϵ as $\epsilon \downarrow 0$ to the solutions of corresponding problems connected with an operator L. For example, if the coefficients of L^ϵ satisfy the Lipschitz condition with the same constant K and converge uniformly to the coefficients of an elliptic operator L, then solutions of certain problems connected with L^ϵ converge to the solutions of corresponding problems for L. If the coefficients of L^ϵ, say, bounded and converge only in the weak sense to the coefficients of L as $\epsilon \downarrow 0$, the solutions of the boundary problems for L^ϵ, in general, will not converge or will converge but not to the

solutions of corresponding problems for L. For example, if

$$L_n^\epsilon = \frac{1}{2}\sum_{i,j=1}^{r} a^{ij}(\frac{x}{\epsilon})\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i-1}^{n} b^i(\frac{x}{\epsilon})\frac{\partial_u}{\partial x^i},$$

and the functions $a^{ij}(x), b^i x$ are 1-periodic in each variable $x^1, \dots, x^n$, the solutions of Dirichlet problem $L^\epsilon u^\epsilon(x) = 0$, $x \in G \subset R^r$, $u^\epsilon(x)|_x = \psi(x)$ converge as $\epsilon \downarrow 0$ to a solution of the same Dirichlet problems for an equation $Lu(x) = 0$. The operator L has constant coefficients but different from the operator with coefficients which are the weak limits of $a^{ij}(\frac{x}{\epsilon})$, $b^i(\frac{x}{\epsilon})$ as $\epsilon \downarrow 0$.

In terms of stochastic processes the problem consists of finding the conditions on the diffusion and drift coefficients providing the weak convergence of the processes. A complete, in a sense, solution of this problem in the one-dimensional case is given in [3]. As has been known since Feller, any one-dimensional diffusion process is defined by a couple of increasing functions $u(x), v(x)$: The operator $D_v d_u f = \frac{d}{dv}(\frac{df}{du})$ is the generator of the process . If X_t^ϵ corresponds to (u^ϵ, v^ϵ) and X_t corresponds to (u, v), then convergence (u_n, v_n) to (u, v) at each continuity point is necessary and sufficient condition for the weak convergence X_t^ϵ to X_t in the space of continuous functions on $[0, T]$, $T < \infty$. The cases of fast oscillating in x periodic and random (invariant with respect to the shifts) coefficients are considered in [3] as applications of the general result. Some other applications to the homogenization problems are considered in [6].

The homogenization problems "on the large deviations level" are considered in [5], [6], [10]. For example, if $\epsilon = 1$ in system (5), and the coefficients and the non-linear terms are 1-periodic in all variables x^i, one can expect that the propagation of the region in R^r where the solution is separated from zero for large t is governed by the Huygens principle. We calculate the asymptotic velocity field which defines the Huygens principle. This velocity field is space homogeneous but, in general, not isotropic. Moreover, it cannot be made isotropic by a linear transformation of the coordinate system. This means that the limiting (for $t \to \infty$) behavior of the front is different from the behavior of the front for equations with space homogeneous coefficients and nonlinear terms. Thus there exist no "effective" equations with constant coefficients which approximate for large t motion of the front in the periodic medium. The homogenization in this problem does not lead to equations with constant coefficients, as in the homogenization problems for "normal deviations". Wave front propagation problems with several small parameters considered in [5], [10] as well.

The work of my graduate student J. Dunyak was supported from that grant. Homogenization problems for reaction-diffusion equations were considered in his thesis [12] and in [13]. The propagation of wave fronts in the periodic media with "holes" is studied there. A number of results concerning homogenization of the boundary conditions included in [12] as well.

4 Asymptotic problems in image reconstruction.

The asymptotic problems for RDE's suggest a number of models of domain growth in R^r, or, in general, of domain time evolution. For example, as was shown in [8], the domain, where the solution is separated from zero, under certain conditions grows according to the Huygens principle. The boundary of the domain can move with the speed proportional to the mean curvature of the boundary under other conditions. The evolution of the domain can be governed by a flow.

Suppose the evolution of a domain is observed in a noise with given statistical characteristics. How can one reconstruct the evolution of the domain? How can one estimate the time when the domain started to grow if the evolution of the domain is observed in a noise?

The first question was considered in [15]. We assume that the evolution of domains $G_t \subset R^2$ is governed by a flow, and we observe $Y^\epsilon(t,x)$, $x \in R^2$, $0 \leq t \leq T$:

$$dY^\epsilon(t,x) = I[G_t]dtdx + \epsilon dW(t,x), \tag{6}$$

Here $I(G)$ is the indicator function of $G \subset R^2$, $W(t,x)$ is the Brownian sheet in R^3.

If $\hat{G}^\epsilon_t$, $0 \leq t \leq T$, is the estimator for the domain evolution G_t, we define the distance between $G_\bullet$ and $G^\epsilon_\bullet$ as the Lebesgue measure of the symmetric difference

$$\ell(\hat{G}^\epsilon, G) = \int_0^t mes(\hat{G}^\epsilon_t \Delta G_t)dt.$$

Close upper and lower bounds for the minimax risk are established in [15] for $0 < \epsilon << 1$.

Suppose now the domain $G_t \subset R^2$ does not change up to an unknown time $\tau, \in [T_0, T]$, and then G_t starts to grow. Let our observation again be defined by equation (6), and we want to estimate the change point. This question is studied on [14] in the small noise asymptotics. Two problems are discussed there: First one deals with the case when the domain's area $S(t)$ suffers a jump at the time τ. The second problem concerns the case when $S(t)$ is changing continuously and just the derivative $\frac{dS(t)}{dt}$ has a jump for $t = \tau$. First, we construct asymptotically optimal in the minimax sense estimators in the case of reduced observations, when just evolution of the area of the domain is observed in the noise. Then we consider the estimators for τ using the whole information about the domain evolution. We discuss also in [14] some change point problems for growing domains when the observations are made in discrete time.

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